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J. Phys. A: Math. Gen. 28 (1995) L287–L293. Printed in the UK

LETTER TO THE EDITOR

More on the q-oscillator algebra and q-orthogonal polynomials

Roberto Floreanini†, Jean LeTourneux‡ and Luc Vinet‡

† Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Dipartimento di Fisica Teorica, Università di Trieste, Strada Costiera 11, 34014 Trieste, Italy

‡ Centre de recherches mathématiques, Université de Montréal, CP 6128, succursale Centre-ville, Montréal, Québec, Canada H3C 3J7

Received 20 March 1995

Abstract. Properties of certain q-orthogonal polynomials are connected to the q-oscillator algebra. The Wall and q-Laguerre polynomials are shown to arise as matrix elements of q-exponentials of the generators in a representation of this algebra. A realization is presented where the continuous q-Hermite polynomials form a basis of the representation space. Various identities are interpreted within this model. In particular, the connection formula between the continuous big q-Hermite and the continuous q-Hermite polynomials is thus obtained, and two generating functions for these last polynomials are derived algebraically.

The q-oscillator algebra [1] is generated by the elements A_+, A_- and $K = q^{-N/2}$ subject to the relations

$$A_-A_+ - \frac{1}{q}A_+A_- = 1 \qquad KA_+ = q^{-1/2}A_+K \qquad KA_- = q^{1/2}A_-K\,. \tag{1}$$

Clearly it represents a q-deformation of the oscillator algebra which is retrieved in the limit $q \to 1$. The algebra (1) has been found to have a number of applications. It was shown, in particular, to provide the algebraic interpretation of various q-special functions. (See, for instance [1–11, 18, 19].) The purpose of the present letter is to present additional results on this topic.

Most special functions have q-analogues [12, 13]; these arise in various connections and, in particular, in the description of systems with quantum group symmetries. The algebraic interpretation of many q-special functions has been shown to proceed in analogy with the Lie theory treatment of their classical, $q \to 1$, counterparts. (See for instance [1, 9, 11].)

One considers q-exponentials of the generators of a q-algebra and observes that their matrix elements in representation spaces are expressible in terms of q-special functions. One then uses models to derive properties of these functions through symmetry techniques. This approach has proven to be very fruitful and is still actively pursued. It is the one we shall adopt here.

The outline of this letter is the following. We shall first introduce the representation of (1) that will be used, and give a realization of this representation where the continuous q-Hermite polynomials play the role of the basis vectors. Matrix elements of q-exponentials of A_+ and A_- will then be evaluated and seen to involve, in some cases, Wall and q-Laguerre polynomials. Finally, these results will be used in conjunction with the model we mentioned

0305-4470/95/100287+07$19.50 © 1995 IOP Publishing Ltd

to obtain formulae involving the continuous q-Hermite polynomials. A connection formula and two generating function identities will thus be derived algebraically.

We shall consider the following representation of the q-oscillator algebra. We shall denote by ξ_n, $n = 0, 1, 2, \ldots$ the basis vectors, and take the generators A_+, A_- and K to act according to

$$A_+\xi_n = -q^{-(n+1)/2}\xi_{n+1} \qquad A_-\xi_n = q^{n/2+1}\left(\frac{1-q^{-n}}{1-q}\right)\xi_{n-1} \qquad K\xi_n = q^{-n/2}\xi_n\,. \tag{2}$$

It is immediately obvious to check that these definitions are compatible with the commutation relations (1). A first connection with q-polynomials is made by observing that there exists a realization of this representation where the continuous q-Hermite polynomials $H_n(x|q)$ appear as basis vectors [14]. These polynomials are defined as follows [13]:

$$H_n(x|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathrm{e}^{\mathrm{i}(n-2k)\theta} \qquad x = \cos\theta \tag{3}$$

with the q-binomial coefficients given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\,. \tag{4}$$

We use standard notation [12, 13] where $(a;q)_\alpha$ stands for

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \qquad |q|<1\,. \tag{5}$$

The classical Hermite polynomials $H_n(x)$ are obtained as follows from the continuous q-Hermite polynomials $H_n(x|q)$ in the limit $q \to 1$:

$$\lim_{q\to 1^-}\left(\frac{1-q}{2}\right)^{-n/2} H_n\left(x\sqrt{\frac{1-q}{2}}\,\middle|\, q\right) = H_n(x)\,. \tag{6}$$

Let T_z be the q-shift operator:

$$T_z f(z) = f(qz)\,. \tag{7}$$

It can be verified [15] that the representation (2) is realized by setting [14]

$$\xi_n(x) = H_n(x|q) \qquad n = 0, 1, 2, \ldots \tag{8}$$

and taking A_+, A_- and K to be the following operators acting on functions of $x = (z+z^{-1})/2$, $z = \mathrm{e}^{\mathrm{i}\theta}$:

$$A_+ = q^{-1/2}\frac{1}{z-z^{-1}}\left(\frac{1}{z^2}T_z^{1/2} - z^2T_z^{-1/2}\right) \tag{9a}$$

$$A_- = \frac{q}{1-q}\frac{1}{z-z^{-1}}\left(T_z^{1/2} - T_z^{-1/2}\right) \tag{9b}$$

$$K = \frac{1}{z-z^{-1}}\left(-\frac{1}{z}T_z^{1/2} + zT_z^{-1/2}\right)\,. \tag{9c}$$

The operator $\tau = (z-z^{-1})^{-1}(T_z^{1/2} - T_z^{-1/2})$ in A_- is referred to as the divided difference operator.

In order to mimic the exponential mapping from Lie algebras to Lie groups, we shall need q-analogues of the function e^x. We introduce the q-exponentials

$$E_q^{(\mu)}(x) = \sum_{n=0}^{\infty}\frac{q^{\mu n^2}}{(q;q)_n}x^n \qquad \mu \in \mathbb{R}\,. \tag{10}$$

In the limit $q \to 1$, once x has been rescaled by $(1-q)$, these functions all tend to the ordinary exponential $\lim_{q\to 1} E_q^{(\mu)}[(1-q)x] = \mathrm{e}^x$. For some specific values of μ, they correspond to standard q-exponentials. Indeed, for $\mu = 0$ and $\mu = \frac{1}{2}$ one has [12]

$$E_q^{(0)}(x) = e_q(x) = \frac{1}{(x;q)_\infty} \tag{11a}$$

$$E_q^{(1/2)}(x) = E_q(q^{-1/2}x) = (-q^{-1/2}x;q)_\infty\,. \tag{11b}$$

Note that $e_q(\lambda x)$ and $E_q(-q\lambda x)$ are, respectively, eigenfunctions with eigenvalue λ of the q-derivative operators $D^+ = z^{-1}(1-T_z)$ and $D^- = z^{-1}(1-T_z^{-1})$. Not so well known is the q-exponential

$$\mathcal{E}_q(x;a,b) = \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q;q)_n} \left(aq^{(1-n)/2}\mathrm{e}^{\mathrm{i}\theta};q\right)_n \left(aq^{(1-n)/2}\mathrm{e}^{-\mathrm{i}\theta};q\right)_n b^n \qquad x = \cos\theta \tag{12}$$

introduced in [16]. It enjoys the property of being an eigenfunction of the divided difference operator τ with eigenvalue $abq^{-1/4}$, and, in the limit $q \to 1$, $\mathcal{E}_q(x;a,(1-q)b) \to \exp[(1+a^2-2ax)b]$. We observe that

$$E_q^{(1/4)}(x) = \mathcal{E}_q(-;0,x)\,. \tag{13}$$

The operators

$$U^{(\mu,\nu)}(\alpha,\beta) = E_q^{(\mu)}\left((1-q)\alpha A_+\right) E_q^{(\nu)}\left(\frac{\beta}{q}(1-q)A_-\right) \tag{14}$$

in the completion of the q-oscillator algebra, are central in our analysis. In the limit $q \to 1$, they go into the Lie group element $\mathrm{e}^{\alpha A_+}\mathrm{e}^{\beta A_-}$. Their matrix elements in the representation space spanned by the vectors ξ_n are defined by

$$U^{(\mu,\nu)}(\alpha,\beta)\xi_n = \sum_{m=0}^{\infty} U_{m,n}^{(\mu,\nu)}(\alpha,\beta)\xi_m \tag{15}$$

and, when evaluated, are found to involve q-special functions. Explicitly, one obtains

$$U_{m,n}^{(\mu,\nu)}(\alpha,\beta) = (-\beta)^{n-m} q^{(n-m)[(\nu+1/4)(n-m)-n/2-1/4]} \begin{bmatrix} n \\ m \end{bmatrix}_q \\ W\mathcal{P}_m^{(\mu,\nu)}\left(-(1-q)\alpha\beta; q^{n-m}|q\right) \qquad \text{if} \quad m \leqslant n \tag{16a}$$

$$U_{m,n}^{(\mu,\nu)}(\alpha,\beta) = \frac{[-(1-q)\alpha]^{m-n}}{(q;q)_{m-n}} q^{(n-m)[(\mu-1/4)(m-n)-n/2-1/4]} \\ W\mathcal{P}_n^{(\nu,\mu)}\left(-(1-q)\alpha\beta; q^{m-n}|q\right) \qquad \text{if} \quad m \geqslant n \tag{16b}$$

where $\mathcal{P}_n^{(\mu,\nu)}(x;q^\gamma|q)$ are the polynomials given by

$$\mathcal{P}_n^{(\mu,\nu)}(x;q^\gamma|q) = \sum_{k=0}^{n} \frac{q^{k^2(\mu+\nu)+2\nu\gamma k}(q^{-n};q)_k}{(q;q)_k(q^{\gamma+1};q)_k} x^k\,. \tag{17}$$

Note that in passing from expression (16a) to (16b) for $U_{m,n}^{(\mu,\nu)}(\alpha,\beta)$ or vice versa, m and n as well as μ and ν, are exchanged in the polynomials $\mathcal{P}_n^{(\mu,\nu)}$.

The connection with standard q-polynomials is observed for particular values of μ and ν. The little q-Laguerre or Wall polynomials [13]

$$p_n(x;a|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix} \,\middle|\, q; qx\right) \tag{18}$$

are, for instance, seen to occur for $\mu = \nu = 0$. Indeed,

$$\mathcal{P}_n^{(0,0)}\big(-(1-q)\alpha\beta; q^{m-n}|q\big) = p_n\big((1-1/q)\alpha\beta; q^{m-n}|q\big). \tag{19}$$

Similarly, the q-Laguerre polynomials [14]

$$L_n^{(\rho)}(x;q) = \frac{(q^{\rho+1};q)_n}{(q;q)_n} \qquad {}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{\rho+1}\end{matrix}\,\middle|\, q; -xq^{n+\rho+1}\right) \tag{20}$$

are found to arise when $\mu = \nu = \frac{1}{4}$. In this case we find, for example,

$$\mathcal{P}_n^{(1/4,1/4)}\big(x; q^{m-n}|q\big) = \frac{(q;q)_n}{(q^{m-n+1};q)_n} L_n^{(m-n)}\big(x; q^{-(m+n+1)/2}; q\big). \tag{21}$$

The q-hypergeometric series ${}_r\phi_s$ that we are using are defined by [12, 13]

$$\begin{aligned} &{}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, & a_r \\ b_1, \ldots, & b_s \end{matrix}\,\middle|\, q; z\right] \\ &\qquad = \sum_{n=0}^{\infty} \frac{(a_1;q)\ldots(a_r;q)_n}{(q;q)_n(b_1,q)_n\ldots(b_s;q)_n}\left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r} z^n. \end{aligned} \tag{22}$$

We shall now return to the model (8) and (9) and make use of these matrix elements to derive properties of the continuous q-Hermite polynomials. We shall first produce a formula giving the expansion of the continuous big q-Hermite polynomials $H_n(x;a|q)$ in terms of continuous q-Hermite polynomials $H_n(x|q)$; that is, we shall derive from symmetry considerations an identity of the form

$$H_n(x;a|q) = \sum_{k=0}^{\infty} C_{k,n} H_k(x|q). \tag{23}$$

The continuous big q-Hermite polynomials are defined by [13]

$$H_n(x;a|q) = e^{in\theta}{}_2\phi_0\left(\begin{matrix} q^{-n}, ae^{i\theta} \\ \text{—} \end{matrix}\,\middle|\, q; q^n e^{-2i\theta}\right) \qquad x = \cos\theta. \tag{24}$$

An algebraic interpretation of these polynomials is found in [17]. It is easy to see from definition (3) that the continuous q-Hermite polynomials $H_n(x|q)$ are the $a \to 0$ limits of the big ones, i.e. $H_n(x|q) = H_n(x;0|q)$. Since the $H_n(x;a|q)$ and *a fortiori* the $H_n(x|q)$ are particular cases of Askey–Wilson polynomials, the coefficients $C_{k,n}$ in (23) can evidently be obtained by specializing the general formula for the connection coefficients of these polynomials [12]. Our purpose here is to show that they can be given an algebraic interpretation.

We adopt the realization (8) and (9) where the basis vectors are the functions $\xi_n(x) = H_n(x|q)$ and the generators the operators (5), and consider within this framework the action of $\mathcal{E}_q(-;0,(\beta(1-q)/q)A_-) = U^{(0,1/4)}(0,\beta)$ on continuous q-Hermite polynomials. Since the computation of the matrix elements of $U^{(\mu,\nu)}(\alpha,\beta)$ is model independent, we have, on the one hand, from (15) and (16*a*):

$$\mathcal{E}_q(-;0,(\beta(1-q)/q)A_-)\, H_n(x|q) = \sum_{k=0}^{n}(-1)^k a^k q^{k(k-1)/2}\begin{bmatrix} n \\ k\end{bmatrix}_q H_{n-k}(x|q) \tag{25}$$

with

$$a = q^{-n/2+1/4}\beta. \tag{26}$$

On the other hand, it turns out that the action of $\mathcal{E}_q(-;0,(\beta(1-q)/q)A_-)$ on $H_n(x|q)$ can be resummed, using the explicit expression (3) and the q-binomial theorem

$$\sum_{n=0}^{\infty}\frac{(\alpha;q)_n}{(q;q)_n}z^n=\frac{(\alpha z;q)_\infty}{(z;q)_\infty}. \tag{27}$$

One arrives at

$$U^{(0,1/4)}(0,\beta)H_n(x|q)=\left(\frac{a}{z};q\right)_n z^n{}_2\phi_1\left(\left.\begin{matrix}q^{-n},0\\ \frac{z}{a}q^{1-n}\end{matrix}\right| q;\frac{q}{az}\right). \tag{28}$$

One then uses the transformation formula [13]

$${}_2\phi_1\left(\left.\begin{matrix}q^{-n},0\\ c\end{matrix}\right| q;z\right)=(-1)^n q^{-n(n+1)/2}\frac{z^n}{(c;q)_n}{}_2\phi_0\left(\left.\begin{matrix}q^{-n},\frac{q^{1-n}}{c}\\ -\end{matrix}\right| q;\frac{q^{2n}c}{z}\right) \tag{29}$$

and definition (26) to show that

$$\mathcal{E}_q(-;0,\beta\tau)H_n(x|q)=H_n(x;a|q) \tag{30}$$

where a is still given by (26). Putting (25) and (30) together finally yields the expansion formula

$$H_n(x;a|q)=\sum_{k=0}^{n}(-1)^k a^k q^{k(k-1)/2}\begin{bmatrix}n\\k\end{bmatrix}_q H_{n-k}(x|q). \tag{31}$$

In the limit $q\to 1$, this relation tends to the following identity between classical Hermite polynomials:

$$H_n(x-a)=\sum_{k=0}^{n}(-1)^{n-k}(2a)^{n-k}\binom{n}{k}H_k(x). \tag{32}$$

This can be verified using (6) and noting that

$$\lim_{q\to 1^-}\left(\frac{1-q}{2}\right)^{n/2}H_n\left(x\sqrt{\frac{1-q}{2}};a\sqrt{2(1-q)}\,\middle|\, q\right)=H_n(x-a). \tag{33}$$

As a second example of application, we shall constructively derive two generating function identities. Consider the action of the operators $U^{(\mu,\nu)}(\alpha/(1-q),0)$ on $\xi_0(x)=1$. We have from (15) and (16*b*)

$$U^{(\mu,0)}(\alpha/(1-q),0)\cdot 1=\sum_{m=0}^{\infty}\frac{(-q^{-1/4}\alpha)^m}{(q;q)_m}q^{(\mu-1/4)m^2}H_m(x|q). \tag{34}$$

On the one hand, this relation tells us how the function $\xi_0(x)=1$ transforms under the action of what would be in the limit $q\to 1$, a transformation group element. On the other hand, if expressions in closed form can be found for $U^{(\mu,0)}(\alpha/(1-q),0)W1$, these would be generating functions for the continuous q-Hermite polynomials.

It is readily seen that two such generating functions can indeed be derived from (34) for $\mu=\frac{1}{4}$ and $\mu=\frac{3}{4}$. In the first case one has, after having set $\mu=\frac{1}{4}$ and inserted the explicit expansion of $H_m(x|q)$ in (34):

$$\mathcal{E}_q(-;0,\alpha A_+)\cdot 1=\sum_{m,k=0}^{\infty}\frac{(-\alpha q^{-1/4})^m}{(q;q)_{m-k}(q;q)_k}z^{m-2k}. \tag{35}$$

When using $\ell=m-k$ instead of m as the summation index, the two sums are seen to split and one finds that

$$\mathcal{E}_q(-;0,\alpha A_+)\cdot 1=e_q\left(-q^{-1/4}\alpha z\right)e_q\left(-q^{-1/4}\frac{\alpha}{z}\right) \tag{36}$$

where $e_q(x)$ is the q-exponential defined in (11a). One now sets $t = -q^{-1/4}\alpha$ to see that (34) entails in the case $\mu = \frac{1}{4}$, the generating function identity

$$e_q(tz)e_q\left(\frac{t}{z}\right) = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} H_n(x|q). \tag{37}$$

The action of $U^{(3/4,0)}(\alpha/(1-q), 0)$ on $\xi_0(x) = 1$ can similarly be cast in closed form. Again using (3) in (34), with $\mu = \frac{3}{4}$, we now have

$$E_q^{3/4}(\alpha A_+) \cdot 1 = \sum_{k,m=0}^{\infty} \frac{q^{m(m-1)/2}}{(q;q)_k (q;q)_{m-k}} \left(-q^{1/4}\alpha\right)^m z^{m-2k}. \tag{38}$$

The two sums are reorganized by using $\ell = m - k$ instead of m as summation index. This allows one to perform the sum over ℓ thanks to the explicit expansion for (11b), and one thus arrives at

$$E_q^{(3/4)}(\alpha A_+) \cdot 1 = \left(q^{1/4}\alpha z; q\right)_\infty {}_1\phi_1 \left(\begin{matrix} 0 \\ q^{1/4}\alpha z \end{matrix} \middle| q; q^{1/4}\frac{\alpha}{z} \right). \tag{39}$$

We set $t = q^{1/4}\alpha$ and combine (38) and (39) to find another generating function identity

$$(tz;q)_\infty {}_1\phi_1 \left(\begin{matrix} 0 \\ tz \end{matrix} \middle| q; \frac{t}{z} \right) = \sum_{k=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} (-1)^n t^n H_n(x|q) \tag{40}$$

the algebraic interpretation of which stems from (34).

The results presented here illustrate once more the usefulness of the algebraic interpretation of q-special functions. It is remarkable that relations (31), (37) and (40), like many other q-special functions identities, have their origin in the representation theory of the q-oscillator algebra.

One of us (LV) greatly benefitted from the support and hospitality extended to him by the Sezione di Trieste of the INFN. The research of JL and LV is supported in part through funds provided by NSERC (Canada) and FCAR (Quebec).

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